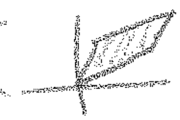
MATH 312 Lecture 17 | MORE DETERMINANTS

Recap:

- Determinants as FUNCTIONS

  det : {n×n matrices} → {numbers}

  satisfying
  - I. Normalization
  - II. Antisymmetry
  - III. Multilinearity

  → Interpretation as ± Volume ()

- Row operations' effect on determinants:
  - Type I : nothing!
  - Type II : change sign
  - Type III : scale determinant.

  det of upper Δ, lower Δ AND diag matrix equals product of diag entries

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TODAY: MORE PROPERTIES, FORMULAS, ETC.

A

(i) By multilinearity, if a row of A is all zeros, then det(A) = 0! (You can just factor out that zero).

(ii) So, if A has two identical rows, then again det(A) = 0 (Just do a TYPE I operation and use (i))

(iii) So if the rows of A are LINEARLY DEPENDENT, then again det(A) = 0 (use row operations to produce a zero row)

(iv) So, if A is $n \times n$, and

$$\left.\begin{cases} \text{Rank}(A) \neq n, \text{ or} \\ \text{RREF}(A) \neq \text{id}, \text{ or} \\ A \text{ is not invertible, or} \\ A \text{ has a nonzero nullspace} \end{cases}\right\} \text{ THEN } \underline{\det(A) = 0}.$$

<u>THIS</u> <u>IS</u> <u>IMPORTANT</u>! If A is not generic, then its determinant must be zero!

B <u>DET</u> OF PRODUCT = PRODUCT OF <u>DETS</u>

- If A & B are $n \times n$, then

$$\boxed{\det(A)\det(B) = \det(AB)}$$

If $\det B = 0$ then both sides are zero! ✓

<u>Proof</u>: If $\det(B) \neq 0$, then define a new function

$$d(A) = \det(AB)/\det(B)$$

And note that it satisfies NAM!

N: $d(I) = \det(B)/\det(B) = 1$

A: flipping rows of A flips rows of AB (but not of B)

M: Scaling a row of A <u>scales</u> the same row of AB, etc.

So, d MUST be det, the only function to satisfy NAM. So,

$$\det(AB) = \det(A)\det(B)$$

CONSEQUENCE

- Here's something NICE about the product formula:

If A is invertible, THEN $A \cdot A^{-1} = $ Identity,

$$\text{So } \underset{\det(A)\cdot\det(A^{-1})}{\underset{\parallel}{\det(AA^{-1})}} = \det(I) = 1$$

$$\text{So } \boxed{\det(A) = 1/\det(A^{-1})}$$

Whenever A is actually invertible.

C TRANSPOSES

Somewhat AMAZINGLY, $\boxed{\det(A^T) = \det(A)}$

for all $n \times n$ matrices A.

To see WHY, use LU-decomposition:

$$A = \underbrace{L}_{\text{lower triangular, 1's on diagonal}}\underbrace{U}_{\text{upper triangular, pivots on diagonal}}$$

By the product formula,

$$\mathcal{U} = \begin{bmatrix} p_1 & & \\ & \ddots & \\ 0 & & p_n \end{bmatrix}$$

$$\det(A) = \det(L) \cdot \det(\mathcal{U})$$
$$= 1 \cdot \det(\mathcal{U}) = \det(\mathcal{U})$$
$$= \text{product of pivots.}$$

But now, $A^T = \mathcal{U}^T L^T$

$$\mathcal{U}^T = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ & & p_n \end{bmatrix}$$

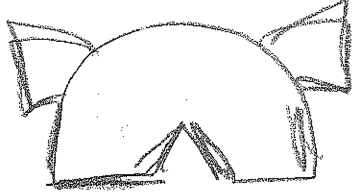
So, $\det(A^T) = \underbrace{\det(\mathcal{U}^T)}_{\text{lower triangular, still has pivots on the diagonal}} \cdot \underbrace{\det(L^T)}_{\text{upper triangular, still has ones on diagonal}}$

$$= \det(\mathcal{U}^T)$$
$$= \text{product of pivots.}$$

--- CONSEQUENCE ---

ANYTHING that was a rule for ROWS of A becomes also a rule for COLUMNS of A.

eg:
- If A has repeated columns, or if A has a zero column, then $\det(A) = 0$
- $\det(A)$ is multilinear in columns!
- If you swap two columns, det changes sign!

C THAT  FORMULA.

Remember this:

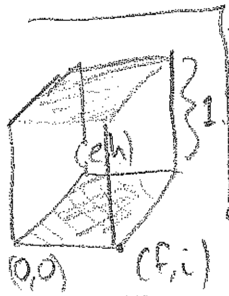
$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} \quad ?$$

This can be derived — EASILY — from the other properties! Look: (multilinearity in first row)

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det\begin{pmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + \det\begin{pmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

And for example, $\det\begin{pmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}$ is the <u>same</u>

as $a \cdot \det\begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}$. Use two TYPE I row operations to get $\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix}$, and finally, $\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$ when we use the <u>volume</u> interpretation.



<u>COFACTORS</u> Here, $\begin{pmatrix} e & f \\ g & i \end{pmatrix}$ is the cofactor matrix corresponding to a. More generally,

<u>if</u> $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Then the (i,j) cofactor matrix is given by deleting i'th row & j'th column.

$$C_{ij} = \begin{array}{cc} i & \begin{bmatrix} a_{11} & & & a_{1n} \\ \hline & & & \\ \hline & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \\ & j \end{array}$$

kill these.

Set $d_{ij} = (-1)^{i+j} C_{ij}$

–ve if $i+j$ is odd
+ve if $i+j$ is even

Then, we have derived.

$$\boxed{\det(A) = a_{11}d_{11} + a_{12}d_{12} + \cdots + a_{1n}d_{1n}}$$

But there is nothing special about the FIRST ROW! We could use the second:

$$\det(A) = a_{21}d_{21} + \cdots + a_{2n}d_{2n}$$

Or the FIRST COLUMN!

$$\det(A) = a_{11}d_{11} + a_{21}d_{21} + \cdots + a_{n1}d_{n1}$$

etc.

The COFACTOR MATRIX is

$$C = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}$$

(Remember! $d_{ij} = (-1)^{i+j} C_{ij}$)

Then, we have shown this: the product

AC^T has $\det(A)$ on each diagonal entry!

BUT the off-diagonal entries are ZERO!

eg (3x3) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $C = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$

$\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} = \det \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ these are NUMBERS

Look at $\begin{bmatrix} d & e & f \end{bmatrix}$ $\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} \end{bmatrix}$

2nd row of A

1st col of C^T

This is zero!

So, $AC^T = \det(A) \, (I)$ identity

And so, if $\det(A) \neq 0$,

$$\boxed{A^{-1} = \frac{C^T}{\det(A)}}$$

Getting inverses from determinants

STANDARD EXAMPLE:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$